At the question period after a Dirac lecture at the University of Toronto, somebody in the audience remarked: "Professor Dirac, I do not understand how you derived the formula on the top left side of the blackboard."
"This is not a question," snapped Dirac, "it is a statement. Next question, please."

- George Gamow, excerpted from *Thirty Years that Shook Physics*, a very fun book on the people involved in the early development of quantum mechanics.

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 1

The eye is focused at infinity, so we assume the rays incident on the eye are all parallel (Fig. 1). From the geometry of the diagram in Fig. 1 it is clear that $\theta = 2\phi$. Snell's law demands that $\sin\theta = n\sin\phi$ where n is the refractive index of the humor. We can make the small angle approximation (making the realistic assumption that light passes only through a small iris in the center of the front of the eye) and just say that $\theta = n\phi$, which gives us $n = 2$.

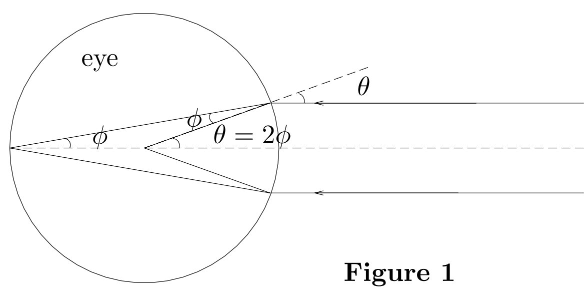

Figure 1

Problem 2

All light coming from the point source is normally incident on the surface of the hemispherical hole in the end of the light guide. Thus the amplitude of the electric field of transmitted light E_t is given by (from Strovink and/or Fowles):

$$\frac{E_t}{E_0} = \frac{2Z_2}{Z_2 + Z_1}, \tag{1}$$

where $Z_{1,2} = \sqrt{\mu_{1,2}/\epsilon_{1,2}}$. Thus the transmission coefficient $t = \frac{E_t}{E_0}$ is given by:

$$t = \frac{2}{1+n}, \tag{2}$$

where $n = 2$ is the refractive index. The percent of light transmitted (intensity) is $T = |t|^2 = 4/9$ in our case.

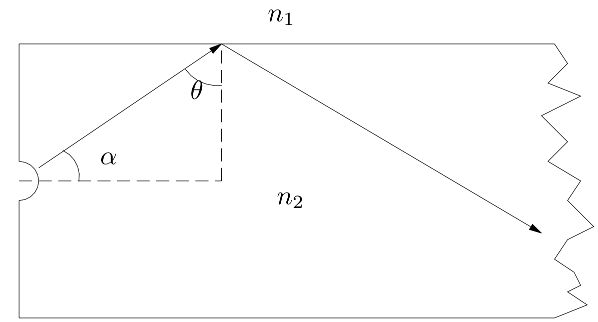

Figure 2

Next consider the diagram in Fig. 2. We require that $\varphi \geq \sin^{-1}(1/n)$ for total internal reflection. From geometry this demands that $\alpha \leq \cos^{-1}(1/n)$. We can now integrate to find the total solid angle $\Delta\Omega$ of light accepted into the light guide:

$$\Delta\Omega = \int_0^{2\pi}\int_0^{\alpha} \sin\theta d\theta d\phi = 2\pi(1 - \cos\alpha) = 2\pi(1 - 1/n) \tag{3}$$

The percent of light accepted is then $\Delta\Omega/4\pi$, or 1/4. So then the fraction of light that travels an appreciable distance is given by the fraction of light transmitted through the interface in the correct direction which is 1/9.

Problem 3

We start out with right-hand circularly polarized light and send it through a quarter-wave plate with the fast axis vertical:

$$\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{4}$$

Next, we send the light through a linear polarizer with the transmission axis at 45^o:

$$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{5}$$

So we have 100% transmission for right-circularly polarized light.

For left circularly polarized light, no light is transmitted:

$$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{6}$$

So the right hand circular analyzer works as claimed.

Problem 4

(a)

If a linear polarizer at 90^o to $\hat{z}$ (the direction of light polarization) is placed in a leg of the Michelson interferometer, no light travels in one leg of the interferometer. Then there will be no fringes and no interference, so $\mathcal{V} \equiv (I_{\max} - I_{\min})/(I_{\max} + I_{\min}) = 0$. This is clear since $I_{\min} = I_{\max}$ if there are no fringes.

(b)

Now, with a linear polarizer at 45^o to $\hat{z}$ upstream of the first linear polarizer, there is light transmitted in both legs of the interferometer. Interference will not occur for light of orthogonal polarizations, so only light polarized in the $\hat{z}$ direction contributes to the fringes.

The linear polarizer at 45^o transmits:

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot E_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{E_0}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{7}$$

The next polarizer only transmits light in the orthogonal direction, so the transmitted light is given by

$$\frac{E_0}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The light bounces off the mirror, which preserves polarization, passes through the second polarizer with no loss of amplitude, then passes through the polarizer at 45^o (which now appears to be at -45^o with respect to the direction of light propagation):

$$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \frac{E_0}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{E_0}{4}\begin{bmatrix} -1 \\ 1 \end{bmatrix}. \tag{8}$$

So now at the output we have two waves from the different arms of the interferometer (supposing light is propagating in the $\hat{x}$ direction):

$$\begin{aligned} \vec{E}_1 &= \frac{E_0}{4}\hat{z} + \frac{E_0}{4}\hat{y} \\ \vec{E}_2 &= E_0 e^{i\phi}\hat{z}, \end{aligned} \tag{9}$$

where ϕ is the phase difference induced by the differing path lengths for the arms of the interferometer.

Thus the intensity of light at the output is given by:

$$I = 2\left(\frac{E_0}{4}\right)^2 + E_0^2 + 2\frac{E_0^2}{4}\cos\phi \tag{10}$$

so we see that for $I_{\max}$ and $I_{\min}$:

$$I_{\max,\min} = \frac{9}{8}E_0^2 \pm \frac{E_0^2}{2}. \tag{11}$$

Using these results in our equation for fringe visibility we find that $\mathcal{V} \equiv (I_{\max} - I_{\min})/(I_{\max} + I_{\min}) = 4/9$

Problem 5

(a)

We have two light beams, whose electric field amplitudes are given by:

$$\mathbf{E}_A \propto \mathrm{Re}[(\hat{x} + i\hat{y})e^{i(kz-\omega t)}]$$

and

$$\mathbf{E}_B \propto \mathrm{Re}[(\hat{x} - i\hat{y})e^{i(kz-\omega t)}]\ ,$$

with Re denoting the real part. We are not given the slit widths, so let's assume that they are small enough to be ignored in our analysis...

With no analyzer in place, the two beams are orthogonally polarized so there is no interference. Thus the intensity at the screen is simply the sum of the two individual intensities, which with small angle approximations is roughly $I \approx 2I_0$.

(b)

With an analyzer that accepts only $\hat{y}$ polarized light, the two light beams have the same polarization after the analyzer and then can interfere. The interference is that of a typical double slit experiment (Young's experiment), as solved in Fowles pp. 59-61. Of course, half the amplitude of each wave has been removed by the analyzer, so the resulting interference pattern is given by:

$$I(Y) = \frac{I_0}{4}\left(1 + \cos\left(\frac{\pi h Y}{\lambda D}\right)\right). \tag{12}$$

(c)

The analyzer blocks out left-hand circularly polarized light so there is contribution only from $\mathbf{E}_B$. Therefore the intensity at the screen is $I \approx I_0$.

Problem 6

We want to prove that

$$\sum_{n=1}^{N} \exp(i\phi_n) = \frac{\sin N\Delta\phi/2}{\sin \Delta\phi/2} \exp(i\bar{\phi}),$$

where

$$\Delta\phi \equiv \phi_{n+1} - \phi_n,$$

and $\bar{\phi}$ is the average of the ϕ_n.

Recall that for a geometric series $\sum_0^\infty ar^n = \frac{a}{1-r}$, where $|r| < 1$. Let us rewrite the sum above as the sum of two infinite series. In order to use the geometric series formula, let's multiply each term in the series by an amplitude $\alpha_n = (1-x)^n$ which ensures that $|\alpha_n \exp(i\phi_n)| < 1$. We can require that the α_n's are so close to unity that they are very well approximated by 1 for the first N terms. We are only worried about the convergence of the tail of the series, which is taken care of with this postulate. We could even take the limit as $x \to 0$ to make this argument more mathematically sound.

Now let's write the series

$$\sum_{n=1}^{N} \exp(i\phi_n)$$

as the difference of two infinite geometric series:

$$\begin{aligned}\sum_{n=1}^{N} \exp(i\phi_n) &\to \sum_{n=0}^{N} \alpha_n \exp(i\phi_1)\exp(in\Delta\phi) \\ &= \sum_{n=0}^{\infty} \alpha_{n+1} \exp(i\phi_1)\exp(in\Delta\phi) - \sum_{n=0}^{\infty} \alpha_{N+n+1} \exp(i\phi_1 + N\Delta\phi)\exp(in\Delta\phi) \\ &= \frac{(1-x)\exp(i\phi_1)}{1-(1-x)\exp(i\Delta\phi)} - \frac{(1-x)^{(N+1)}\exp(i\phi_1 + N\Delta\phi)}{1-(1-x)\exp(i\Delta\phi)}\end{aligned} \tag{13}$$

We now let x go to 0, and we then have:

$$\begin{aligned}\sum_{n=1}^{N} \exp(i\phi_n) &= \frac{\exp(i\phi_1)}{1-\exp(i\Delta\phi)} - \frac{\exp(i\phi_1 + N\Delta\phi)}{1-\exp(i\Delta\phi)} \\ &= \exp\left(\phi_1 + \frac{N-1}{2}\Delta\phi\right)\left(\frac{\exp(-iN\Delta\phi/2) - \exp(+iN\Delta\phi/2)}{\exp(-i\Delta\phi/2) - \exp(+i\Delta\phi/2)}\right)\end{aligned} \tag{14}$$

from which we deduce that:

$$\sum_{n=1}^{N} \exp(i\phi_n) = \frac{\sin N\Delta\phi/2}{\sin \Delta\phi/2} \exp(i\bar{\phi}) \tag{15}$$

proving the original conjecture.

Problem 7

As discussed in Fowles pp. 126-128, this problem can be solved using Fresnel zones. The radius of the Nth Fresnel zone in this case is given by:

$$R = \sqrt{N\lambda D} \tag{16}$$

since the factor $L = (1/h + 1/h')^{-1} = D$ in our case (see Fowles Eq. 5.36).

(a)

(C) The smallest radius for which the light from Fresnel zones cancel is that which includes the first two. In this case the optical disturbance is given by

$$U_p = |U_1| - |U_2| \approx 0.$$

(A) With no screen, the optical disturbance is half that due to the first Fresnel zone,

$$U_p = \frac{1}{2}|U_1|.$$

(B) If we block out the first two Fresnel zones, the optical disturbance is approximately half that due to the third zone, or

$$U_p = \frac{1}{2}|U_3| \approx \frac{1}{2}|U_1|.$$

So as you can see, the choice of

$$R = \sqrt{2\lambda D}$$

satisfies all the required conditions.

(b)

This will hold whenever a similar situation occurs, i.e. an even number of Fresnel zones are blocked by the black disk. The optical disturbances U_A and U_B will always sum to equal the optical disturbance without the screen U_0 because they are complementary apertures. However, only when either U_A or U_B is zero can the squares ($\propto I$) be equal.

So whenever $R = \sqrt{2n\lambda D}$ where n is an integer, this is the case.

Problem 8

From Maxwell's equations in conducting media, we get the wave equations:

$$\begin{aligned} \nabla^2 \vec{E}(z,t) &= \tilde{E}_0 e^{i(\kappa z - \omega t)} \\ \nabla^2 \vec{B}(z,t) &= \tilde{B}_0 e^{i(\kappa z - \omega t)} \end{aligned} \tag{17}$$

where

$$\kappa^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega. \tag{18}$$

For a good conductor,

$$\kappa = \sqrt{\frac{\mu\sigma\omega}{2}}(1+i). \tag{19}$$

Also from Maxwell's equations for a conducting medium, we find that:

$$B_0 = \frac{|\kappa|}{\omega}. \tag{20}$$

The power lost per square meter due to ohmic heating is given by the relation:

$$\begin{aligned} \langle P \rangle &= \sigma \int \langle E^2 \rangle dV \\ &= \frac{\sigma}{2} \int |\tilde{E}_0|^2 \exp{(-\sqrt{\frac{\mu\sigma\omega}{2}} z)} = \frac{\sigma}{2}\sqrt{\frac{2}{\mu\sigma\omega}} E_0^2 \end{aligned} \tag{21}$$

The average value of $|\vec{S}|$ is given by $\langle \frac{1}{\mu} \vec{E} \times \vec{B} \rangle$. Plugging in values from above, we find

$$\langle |\vec{S}| \rangle = \frac{\sigma}{2}\sqrt{\frac{2}{\mu\sigma\omega}} E_0^2 \tag{22}$$

which confirms the conjecture stated in the problem.